Omega Class #2 – Conic Sections

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Introduction to conic sections

Conic sections are planar curves which result from intersecting a cone with a plane in space. They have a variety of surprising properties which make them ubiquitous in optics, acoustics, and celestial mechanics.

Imagine an infinitely large cone oriented point-up in space, and imagine a plane cutting through it. When the plane is horizontal, the intersection is a **circle**, which is the simplest type of conic section. As the plane is tilted, the intersection becomes an **ellipse** with gradually increasing eccentricity. Once the plane reaches exactly the same level of incline as the side of the cone itself, the intersection becomes a **parabola**. And when the plane is more inclined than the side of the cone, the intersection is one of the two halves of a **hyperbola**.[1]

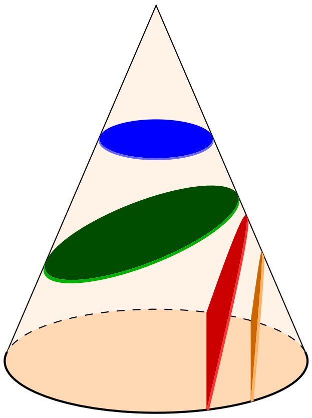

Figure 1: The four types of conic sections, with their interiors shaded in.

All types of conic sections can be defined in the two-dimensional plane, without any recourse to three-dimensional geometry. We'll start there.

Parabolas and other geometric objects

Many familiar ingredients from geometry can be defined as the set of points satisfying a certain logical property.[2] For example,

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[1] The other half of the hyperbola can be recovered by adding a second upside-down cone above the first, with the points touching each other.

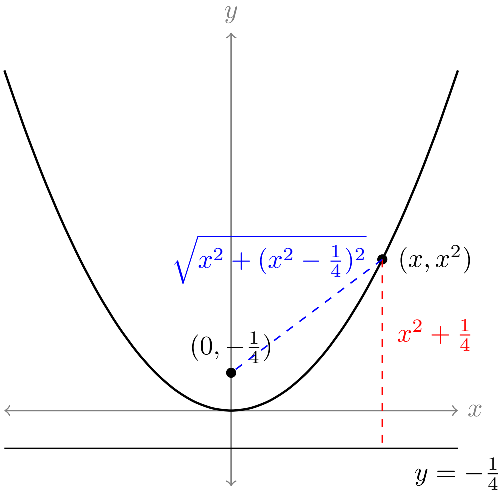
[2] Such a set of points in geometry is called a *locus* of points.

- Given two points P and Q, the set of points equidistant from P and Q is a **line** – more precisely, it is the perpendicular bisectors of PQ. What this means is that every point on this line has equal distance from P and Q, and every point of equal distance from P and Q is on this line.
- Given a point P and a distance r, the set of points which are distance r away from P is a **circle** – where P is called the *center* and r is called the *radius*. Every point of distance r from P lies on this circle, and every point on this circle is distance r from P.

A parabola can be defined in a similar spirit. Given a point P and a line L, the set of points equidistant from P and from L is called a **parabola**. The point P is called the *focus* and the line L is called the *directrix*.

You've likely seen parabolas defined in a different context: for example, as the trajectory traced by a thrown object, or equivalently as the graph of a function $y = ax^2$ for any constant a. We'll prove that these are related.

Proposition: The parabola in the xy-plane with focus $(0, \frac{1}{4})$ and directrix $y = -\frac{1}{4}$ is the graph of the relationship $y = x^2$.



Proof: Let $Q = (x, y)$ be any point on the parabola. By definition, Q is equidistant from the focus $P = (0, \frac{1}{4})$ and the line $L = (y = -\frac{1}{4})$. We can express these distances algebraically:

$$d(P, Q) = \sqrt{x^2 + (y - \frac{1}{4})^2} \qquad\qquad d(P, L) = y + \frac{1}{4}$$

Therefore,

$$\begin{aligned}
\sqrt{x^2+(y-\frac{1}{4})^2} &= y+\frac{1}{4} && (1)\\
x^2+(y-\frac{1}{4})^2 &= (y+\frac{1}{4})^2 && (2)\\
x^2 &= (y+\frac{1}{4})^2-(y-\frac{1}{4})^2 && (3)\\
&= y^2+\frac{y}{2}+\frac{1}{16}-(y^2-\frac{y}{2}+\frac{1}{16}) && (4)\\
&= y && (5)
\end{aligned}$$

so $y = x^2$. Therefore, any point (x, y) on the parabola also satisfies the relationship $y = x^2$. In the other direction, suppose that $Q = (x, y)$ is a point satisfying the relationship $y = x^2$. Then we have

$$\begin{aligned}
x^2 &= (y+\frac{1}{4})^2-(y-\frac{1}{4})^2 && (6)\\
x^2+(y-\frac{1}{4})^2 &= (y+\frac{1}{4})^2 && (7)\\
\sqrt{x^2+(y-\frac{1}{4})^2} &= y+\frac{1}{4} && (8)\\
d(Q,P) &= d(Q,L) && (9)
\end{aligned}$$

which implies that Q lies on the parabola with focus P and directrix L. ■

The argument above is an example of a mathematical **proof**, where every assertion is precisely stated and follows rigorously from the one before. Notice that we proved both "directions" above. This is important because the first part of the proof alone implies that the parabola with focus P and directrix L is a *subset* of the graph of $y = x^2$, but not that they are the same![3] Mathematical proofs are used to demonstrate that an assertion is definitively true via a logically airtight argument. Most communication in and about mathematics is not done by proofs, because they can be stifling, but they are a useful tool to ensure correctness.

Exercise: Prove that the graph of $y = ax^2$ is a parabola with focus $(0, \frac{1}{4a})$ and directrix $y = -\frac{1}{4a}$. (This is part of the homework.)

Exercise: Prove that the graph of $y = ax^2 + bx + c$ is a parabola, and find its focus and directrix.

Lastly, note that not every parabola in the plane satisfies an equation of the form $y = ax^2 + bx + c$. These are only the parabolas whose directrix is horizontal. When we discuss rotations in the plane, we will be able to write down a general formula for a parabola.

Optical properties of parabolas

Parabolas satisfy a third characterization, which makes them extremely relevant for optics. It is a combination of two simultaneous properties (which you'll be prompted to explore on the homework):

- A ray emanating from the focus in any direction will bounce off of the parabola going in a

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[3] For comparison, you can show that every point on the graph of $y = \sqrt{1-x^2}$ lies on the circle with center $(0, 0)$ and radius 1, but you cannot show that every point on the circle lies on the graph of $y = \sqrt{1-x^2}$, because they are not actually the same set of points: the graph of $y = \sqrt{1-x^2}$ is only half of the circle.

trajectory perpendicular to the directrix. (When a ray hits a mirror, the angle of the incoming trajectory and the outgoing trajectory make the same angle with the line tangent to the mirror at the point of impact.)

- Two different rays sent out from the focus in different directions at the same speed, after bouncing off the mirror, end up at exactly the same distance away from the directrix.

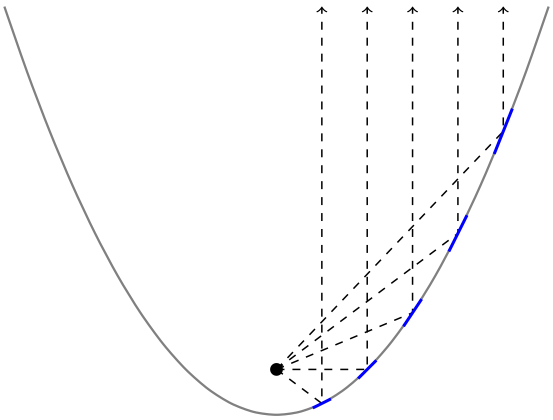

Figure 2: The five dashed trajectories shown all have the same total length. The tangent lines to the parabola at the points of incidence are shown in blue.

In optics, these two properties together imply that a parabolic mirror converts a spherical wave emanating from the focus, into a plane wave traveling orthogonal to the directrix, and vice versa. This is exactly why parabolic mirrors have the highest possible fidelity when used in reflecting telescopes, headlights, directional microphones, etc.

Not only does a parabola satisfy these two properties, but it's also true that any curve satisfying this property is a parabola! Stated more precisely: if $\mathcal{C}$ is a smooth, simple curve in the plane, and there is a point P such that any outgoing ray from P, it

1. intersects $\mathcal{C}$ once,
2. reflects off in a vertical upward trajectory, and
3. for any two such rays emitted at the same time and at the same speed, they are eventually at the same y-coordinate,

then $\mathcal{C}$ is a parabola and P is its focus. Proving this statement requires some calculus, so we won't show it here.