Omega HW #4 – Geometric sequences and series

Krishanu Sankar

November 2, 2025

A *geometric sequence* with ratio r is a sequence of numbers $a_0, a_1, a_2, \ldots$ where each term is r times the previous term, i.e. $a_n = ra_{n-1}$. If the first term is equal to c, then the n-th term is equal to cr^{n-1}. A *geometric series* is the sum of the terms in a geometric sequence. We proved the formula for a *finite* geometric series

$$c + cr + cr^2 + cr^3 + \ldots + cr^{n-1} = \frac{c(1 - r^n)}{1 - r}$$

We also proved that when $|r| < 1$, the *infinite* geometric series has sum

$$c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1 - r}$$

1. (1) Here you calculate a few geometric series sums.
   (a) Calculate the infinite geometric series $5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \ldots$.
   (b) Consider the geometric sequence whose zero-th and first terms are $a_0 = 2, a_1 = 3, \ldots$. Write a formula for the n-th term of the sequence. Then use this formula to calculate the sum of the first 5 terms.

2. (2) Show each of the following equalities by writing the decimal number on the left side as a geometric series, and then applying the formula for the sum of an infinite geometric series.

$$0.333333\ldots = 1/3$$

$$0.111111\ldots = 1/9$$

$$0.090909\ldots = 1/11$$

$$0.142857142857\ldots = 1/7$$

3. (1) Consider the infinite geometric series formula $c + cr + cr^2 + \ldots = \frac{c}{1-r}$. What would happen on both sides if $r = 1$ or $r = -1$? What if $|r| > 1$? Explain in a few sentences.

4. (2) The relationships $a_n = ra_{n-1}$ defining a geometric series is called a *linear recurrence relation* of order 1. In this question we study linear recurrence relations of order 2, i.e. $a_n = ra_{n-1} + sa_{n-2}$. We focus on $a_n = 5a_{n-1} - 6a_{n-2}$.
   (a) Show that the geometric sequence $a_0 = 2, a_1 = 6, a_2 = 18, a_3 = 54\ldots$ satisfies the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$.
   (b) Show that the geometric sequence $b_0 = 1, b_1 = 2, b_2 = 4, b_3 = 8\ldots$ also satisfies the same recurrence relation $b_n = 5b_{n-1} - 6b_{n-2}$.
   (c) Show that if we add these two geometric sequences term-wise to obtain the new sequence $c_0 = 3, c_1 = 8, c_2 = 22, c_3 = 62, \ldots$, then $c_n = 5c_{n-1} - 6c_{n-2}$.

(d) A sequence $a_0, a_1, a_2, a_3, \ldots$ is known to be a geometric sequence with some unknown ratio r, and is also known to satisfy the relationship $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \geq 2$. What could the possible values of r be? Why?

(e) Can you write down another sequence $d_0, d_1, d_2, \ldots$ different from the one in part (c) which is *not* a geometric sequence, but still satisfies the recurrence $d_n = 5d_{n-1} - 6d_{n-2}$? Can you describe a method to make new such sequences?

(f) A sequence $d_0, d_1, d_2, \ldots$ is defined by $d_0 = 0$, $d_1 = 1$, and $d_n = 5d_{n-1} - 6d_{n-2}$ for all $n \geq 2$. Find a general formula for the n-th term of this sequence.

6. (2) In this question, we look at infinite series whose terms are formed by multiplying a geometric sequence by another sequence. Let r be a number with $|r| < 1$.

(a) First we search for a formula for the infinite sum $1 + 2r + 3r^2 + 4r^3 + \ldots$. Show that

$$(1 + r + r^2 + r^3 + \ldots)^2 = 1 + 2r + 3r^2 + 4r^3 + \ldots$$

(b) Use this to calculate the value of the sum $1 + 2(1/3) + 3(1/3)^2 + 4(1/3)^3 + \ldots$.

(c) Next, we search for a formula for the infinite sum $1 + 4r + 9r^2 + 16r^3 + \ldots$. Show that

$$(1 + r + r^2 + r^3 + \ldots)(1 + 3r + 5r^2 + 7r^3 + \ldots) = 1 + 4r + 9r^2 + 16r^3 + \ldots$$

(d) Use part (a) to find a formula for $1 + 3r + 5r^2 + \ldots$, and then use this to calculate

$$1 + 4(1/5) + 9(1/5)^2 + 16(1/5)^3 + 25(1/5)^4 + \ldots$$

Bonus – Calculating square roots

7. (3) Remember the iterative method of calculating $\sqrt{a}$ discussed in Class #1: we begin with an initial guess x_0, and then we repeatedly apply the function $f(x) = \frac{1}{2}(x + \frac{a}{x})$ to get better approximations $x_1, x_2, x_3, \ldots$. In this question you'll analyze this method using the geometric series formula. For simplicity we'll set $a = 2$, so that the function is $f(x) = \frac{1}{2}(x + \frac{2}{x})$.

(a) Let $x = (1 + e)\sqrt{2}$ for some number e with $|e| < 1$. Show that $\frac{2}{x} = (1 - e + e^2 - \ldots)\sqrt{2}$.

(b) Use part (a) to show that $f(x) = (1 + \frac{e^2}{2(1+e)})\sqrt{2}$.

(c) Suppose that x is a fairly good approximation to $\sqrt{2}$ – e.g. $0.9\sqrt{2} < x < 1.1\sqrt{2}$. Show that $\sqrt{2} < f(x) < 1.01\sqrt{2}$.

(d) Suppose that x is a very good approximation to $\sqrt{2}$ – e.g. $0.99\sqrt{2} < x < 1.01\sqrt{2}$. Show that $\sqrt{2} < f(x) < 1.0001\sqrt{2}$.

(e) Suppose that $x \approx \sqrt{2}$ is correct to d digits of accuracy, i.e., $(1 - \frac{1}{10^d})\sqrt{2} < x < (1 + \frac{1}{10^d})\sqrt{2}$. Show that $f(x) \approx \sqrt{2}$ is correct to about $2d$ digits of accuracy, i.e.

$$\sqrt{2} < f(x) < (1 + \frac{1}{10^{2d}})\sqrt{2}$$

(f) Suppose we begin with the initial approximation $x_0 = 1.5$, which satisfies $x_0 < 1.1\sqrt{2}$ and apply the iterative method described. By the reasoning described above, at least how many digits of accuracy will x_4 have?