

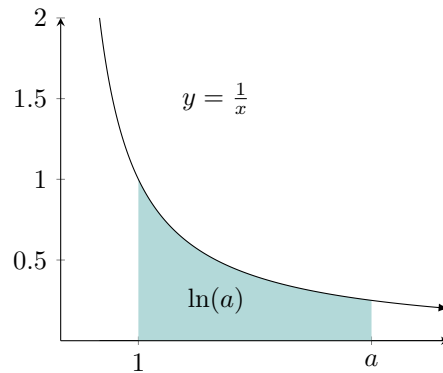
# Omega HW #5 – Logarithms

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## Logarithms

In class, we defined the *natural logarithm* of  $a$ , for any  $a > 0$ , as the area under the curve  $y = \frac{1}{x}$  from  $x = 1$  to  $x = a$ .



We showed that  $\ln(ab) = \ln(a) + \ln(b)$  for any  $a$  and  $b$ . We then defined the logarithm base- $b$  by

$$\log_b(a) = \frac{\ln(a)}{\ln(b)}$$

1. (1) Show the following facts, using the definition of the base- $b$  logarithm above.

(a)  $\log_3(81) = 4$

**Solution:** Because  $81 = 3^4$ , we have

$$\ln(81) = \ln(3 \cdot 3 \cdot 3 \cdot 3) = \ln(3) + \ln(3) + \ln(3) + \ln(3) = 4\ln(3)$$

Therefore,  $\log_3(81) = \frac{\ln(81)}{\ln(3)} = 4$ .

(b)  $\log_a(b) \cdot \log_b(c) = \log_a(c)$

**Solution:**  $\log_a(b) = \frac{\ln(b)}{\ln(a)}$  and  $\log_b(c) = \frac{\ln(c)}{\ln(b)}$ , so

$$\log_a(b) \cdot \log_b(c) = \frac{\ln(b)}{\ln(a)} \cdot \frac{\ln(c)}{\ln(b)} = \frac{\ln(c)}{\ln(a)} = \log_a(c)$$

2. (1) Calculate the following without a calculator:

(a)  $\log_2(56) - \log_2(7)$

**Solution:**

$$\log_2(56) - \log_2(7) = \log_2(56/7) = \log_2(8) = 3$$

(b)  $5^{2 \log_5(6)}$

**Solution:** The answer is 36. Two different ways to calculate this:

$$5^{2 \log_5(6)} = 5^{\log_5(6^2)} = 5^{\log_5(36)} = 36$$

$$5^{2 \log_5(6)} = (5^{\log_5(6)})^2 = 6^2 = 36$$

(c)  $3 \log_{10}(200) - \frac{1}{2} \log_{10}(64)$

**Solution:** We separately calculate that

$$\frac{1}{2} \log_{10}(64) = \log_{10}(64^{1/2}) = \log_{10}(8)$$

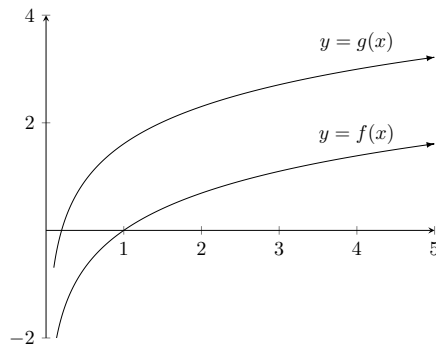
$$3 \log_{10}(200) = \log_{10}(200^3) = \log_{10}(8000000)$$

Therefore,

$$3 \log_{10}(200) - \frac{1}{2} \log_{10}(64) = \log_{10}(8000000) - \log_{10}(8) = \log_{10}(10^6) = 6$$

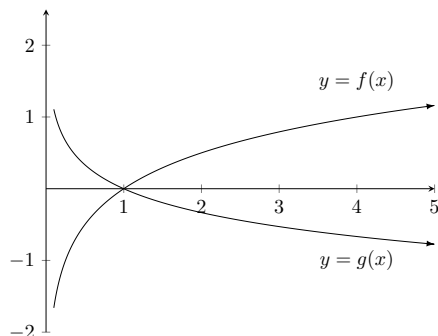
3. (2) In this question, you will be asked to describe the relationship among various logarithmic functions.

- (a) Let  $f(x) = \ln(x)$  and  $g(x) = \ln(5x)$ . What is the relationship between these two functions? Show that  $g(x) = f(x) + C$  for some constant  $C$ . What is this constant?



**Solution:** First, we notice that  $g(x) = f(5x)$ , which means that the graph of  $y = f(x)$  is a **horizontal stretch** of the graph of  $y = g(x)$  by a factor of 5. We also see that  $\ln(5x) = \ln(x) + \ln(5)$ , so  $g(x) = f(x) + \ln(5)$ . So the graph of  $y = g(x)$  is an **upward translation** of the graph of  $y = f(x)$  by  $\ln(5)$ . Both of these are true!

- (b) Let  $f(x) = \log_{1/8}(x)$  and  $g(x) = \log_4(x)$ . Show that  $g(x) = Cf(x)$  for some constant  $C$ . What is this constant?



**Solution:** We calculate that  $\ln(4) = \ln(2^2) = 2\ln(2)$  and  $\ln(1/8) = \ln(2^{-3}) = -3\ln(2)$ . Therefore  $\ln(4) = -\frac{2}{3} \cdot \ln(1/8)$ . So,

$$\log_4(x) = \frac{\ln(x)}{\ln(4)} = \frac{\ln(x)}{-\frac{2}{3}\ln(1/8)} = -\frac{3}{2} \cdot \frac{\ln(x)}{\ln(1/8)} = -\frac{3}{2} \log_{1/8}(x)$$

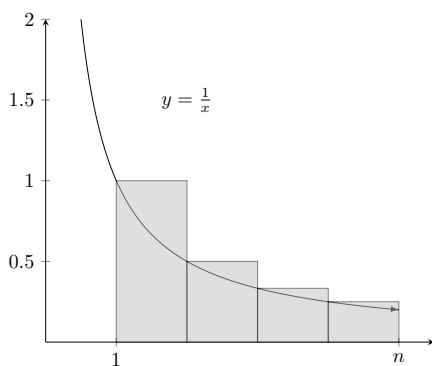
i.e.  $g(x) = -\frac{3}{2}f(x)$ .

4. (2) Consider the **harmonic sequence**

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

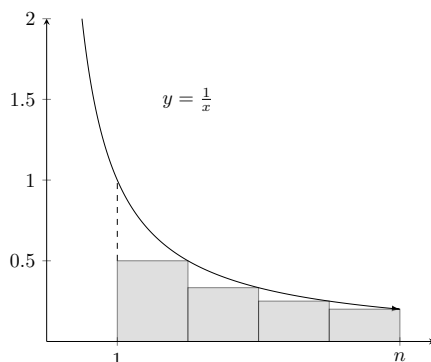
Let  $H_n$  denote the sum of the first  $n$  terms, namely  $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ . We will study the size of  $H_n$  as  $n$  grows.

(a) Show that  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} > \ln(n)$ , using the following picture.



**Solution:** Remember that  $\ln(n)$  is the area under the hyperbola from  $x = 1$  to  $x = n$ , which breaks up into  $n-1$  pieces: the area from  $x = 1$  to  $x = 2$ , the area from  $x = 2$  to  $x = 3$ , the area from  $x = 3$  to  $x = 4$ , etc. The area under the graph of  $y = 1/x$  from  $x = 1$  to  $x = 2$  is less than the area of the leftmost rectangle, which has height 1 and width 1. The area from  $x = 2$  to  $x = 3$  is less than the area of the second rectangle, which has height  $1/2$  and width 1. The area from  $x = 3$  to  $x = 4$  is less than the area of the third rectangle, which has height  $1/3$  and width 1. Continuing on, we deduce that the sum of these  $n - 1$  pieces is less than  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}$ , which exactly tells us that  $\ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}$ .

- (b) Use a similar argument to show that  $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < \ln(n)$ .



**Solution:** This time, we lower-bound the areas under the graph of the hyperbola by rectangles. The area from  $x = 1$  to  $x = 2$  is greater than the first rectangle, which has height  $1/2$ . The area from  $x = 2$  to  $x = 3$  is greater than the second rectangle, which has height  $1/3$ . Etc. Summing these up, we find that the area under the hyperbola from  $x = 1$  to  $x = n$ , which itself is equal to  $\ln(n)$  is greater than  $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ .

- (c) Using the previous two parts, show that  $\ln(n+1) < H_n < \ln(n) + 1$ .

**Solution:** Since  $H_{n-1}$  is defined as  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}$ , part (a) tells us that  $H_{n-1} > \ln(n)$  for all  $n$ . This means that  $H_n > \ln(n+1)$  for all  $n$ . In part (b), we have  $H_n - 1 < \ln(n)$ , which means  $H_n < \ln(n) + 1$ . Putting these together, we learn that for every value of  $n$ ,

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < \ln(n) + 1$$

- (d) Does the infinite sum  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  converge to a finite value? Why or why not?

**Solution:** It does not. Consider the sum  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$  where  $n$  is one less than a power of 2, i.e.  $n = 2^k - 1$  for some  $k$ . Then

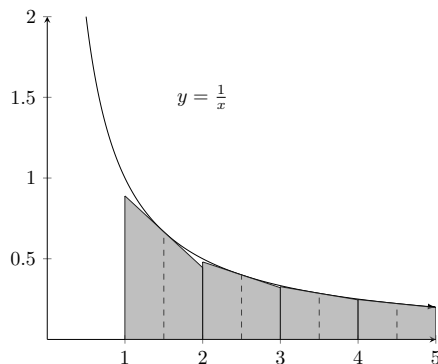
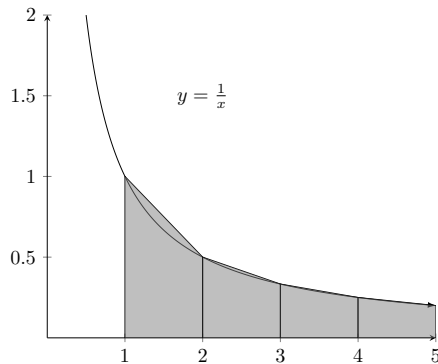
$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k - 1} > \ln(2^k) = k \ln(2)$$

As we let  $k$  become larger and larger, the number  $k \ln(2)$  goes to infinity. Therefore, the infinite series sum  $1 + \frac{1}{2} + \frac{1}{3} + \dots$  goes to infinity.

- (e) **(Challenge!)** Use a similar style of argument to show the following two tighter inequalities:

$$\frac{1 + 1/2}{2} + \frac{1/2 + 1/3}{2} + \frac{1/3 + 1/4}{2} + \dots + \frac{1/(n-1) + 1/n}{2} > \ln(n)$$

$$\frac{2}{1+2} + \frac{2}{2+3} + \frac{2}{3+4} + \dots + \frac{2}{(n-1)+n} < \ln(n)$$

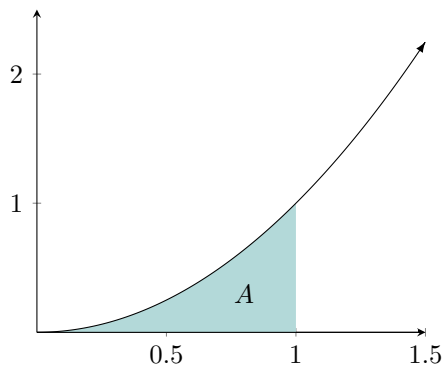


**Solution:** The first figure shown sets an upper bound on the area from  $x = 1$  to  $x = n$  using trapezoids whose vertices are at  $(1, 1)$ ,  $(2, 1/2)$ ,  $(3, 1/3)$ ,  $\dots$ . The area of the first trapezoid is equal to the width (which is 1) times the average of the two bases (which are 1 and  $1/2$ ). The area of the second trapezoid is equal to the width (which is 1) times the average of the two bases (which are  $1/2$  and  $1/3$ ). The area of the third trapezoid is equal to the width (which is 1) times the average of the two bases (which are  $1/3$  and  $1/4$ ). Etc. Summing these up implies that the total area of the trapezoids shown is equal to  $\frac{1+1/2}{2} + \frac{1/2+1/3}{2} + \frac{1/3+1/4}{2} + \dots + \frac{1/(n-1)+1/n}{2}$ .

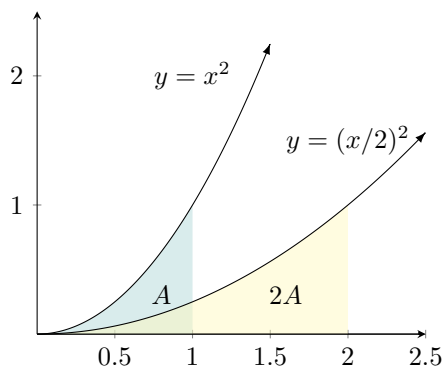
The second figure shown sets a lower bound on the area from  $x = 1$  to  $x = n$  using trapezoids which are tangent to the graph at  $(1.5, \frac{1}{1.5})$ ,  $(2.5, \frac{1}{2.5})$ ,  $(3.5, \frac{1}{3.5})$ ,  $\dots$ . The area of the first trapezoid is equal to the width (which is 1) times the average of the two bases, which is  $\frac{1}{1.5}$ . The area of the second trapezoid is equal to the width (which is 1) times the average of the two bases, which is  $\frac{1}{2.5}$ . Etc. Summing these up implies that the total area of the trapezoids is equal to  $\frac{1}{1.5} + \frac{1}{2.5} + \dots + \frac{1}{n-0.5}$  which is equivalent to the second desired inequality.

## Quadrature of a Parabola

5. (2) In this problem, you'll be guided through the calculation of the area under the graph of the parabola  $y = x^2$  from  $x = 0$  to  $x = 1$ , labeled below as  $A$ .

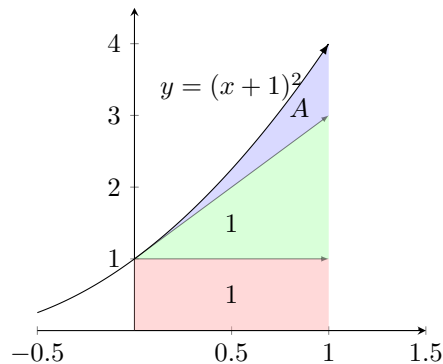


- (a) Argue geometrically that the area under the graph of  $y = (x/2)^2$  from  $x = 0$  to  $x = 2$  equals  $2A$ . Then argue geometrically that the area under the graph of  $y = x^2$  from  $x = 0$  to  $x = 2$  equals  $8A$ .



**Solution:** The graph of  $y = (x/2)^2$  is a factor-of-two horizontal stretch of the graph of  $y = x^2$ . Therefore, the area under the graph of  $y = (x/2)^2$  from  $x = 0$  to  $x = 2$  is a factor-of-two horizontal stretch of the area  $A$ , and therefore has value  $2A$ . Next up, the graph of  $y = x^2$  is a factor-of-four vertical stretch of the graph of  $y = (x/2)^2 = x^2/4$ , and therefore the area under the graph of  $y = x^2$  from  $x = 0$  to  $x = 2$  has area  $4 \cdot 2A = 8A$ .

- (b) Shift the graph to the left by 1 to get  $y = (x + 1)^2 = x^2 + 2x + 1$ . Show that the area under the graph of  $y = 1$  from  $x = 0$  to  $x = 1$  is equal to 1, and also that the area under the graph of  $y = 2x$  from  $x = 0$  to  $x = 1$  is also equal to 1. Argue that therefore, the area under the graph of  $y = x^2 + 2x + 1$  from  $x = 0$  to  $x = 1$  is equal to  $A + 1 + 1$ .



**Solution:** The area under the graph of  $y = 1$  from  $x = 0$  to  $x = 1$  is just a square of sidelength 1, so it has area 1. The area under the graph of  $y = 2x$  from  $x = 0$  to  $x = 1$  is a triangle with width 1 and height 2, so it has area  $\frac{1 \cdot 2}{2} = 1$ . So the area under the graph of  $y = x^2 + 2x + 1$  from  $x = 0$  to  $x = 1$  is equal to the **sum** of the corresponding areas for the graphs of  $y = x^2$ ,  $y = 2x$ , and  $y = 1$ , which are equal to  $A$ , 1, and 1 respectively.

(c) Combine the two calculations above to calculate the value of  $A$ .

**Solution:** In part (b), we found that the area under the graph of  $y = (x + 1)^2$  from  $x = 0$  to  $x = 1$  is equal to  $A + 2$ . This second area is equal to the area under the graph of  $y = x^2$  from  $x = 1$  to  $x = 2$ , by a horizontal translation. Therefore, the area under the graph of  $y = x^2$  from  $x = 0$  to  $x = 2$  is equal to  $A + A + 2 = 2A + 2$ . But in part (a), we found that the area under the graph of  $y = x^2$  from  $x = 0$  to  $x = 2$  is equal to  $8A$ . Therefore,  $2A + 2 = 8A \implies 6A = 2 \implies \boxed{A = 1/3}$ .

6. (3) **Challenge:** Can you use the same technique to calculate the area under the graph of  $y = x^3$  from  $x = 0$  to  $x = 1$ ? What about  $y = x^4$ ? If you have done these correctly, you should spot a pattern: can you conjecture a formula for the area under the graph of  $x^n$  from  $x = 0$  to  $x = 1$ ?

**Solution:** Let  $B$  denote the area under the graph of  $y = x^3$  from  $x = 0$  to  $x = 1$ . Mimicking part (a), you can show that the area under the graph of  $y = x^3$  from  $x = 0$  to  $x = 2$  is equal to  $16B$ . And mimicking part (b), the area under the graph of  $y = (x + 1)^3 = x^3 + 3x^2 + 3x + 1$  from  $x = 0$  to  $x = 1$  is equal to  $B + 3A + 3/2 + 1$ . Therefore,

$$16B = B + B + 3A + \frac{3}{2} + 1 = 2B + 3(1/3) + \frac{5}{2} = 2B + 7/2$$

$$\implies 14B = 7/2 \implies \boxed{B = 1/4}$$

If you do the same for  $x^4$ , calling that area  $C$ , you should get

$$32C = C + C + 4B + 6A + \frac{4}{2} + 1 = 2C + 6 \implies \boxed{C = 1/5}$$

(More generally, the area under the graph of  $x^n$  from  $x = 0$  to  $x = 1$  is equal to  $\frac{1}{n+1}$ .)